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Polynomial Approximation of Generalized Biaxisymmetric Potentials*

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1. INTRODUCTION

The real valued generalized biaxisymmetric potentials (GBASP) $F^{(\alpha,\beta)}$ regular in the open unit hypersphere $\Sigma^{(\alpha,\beta)}$ about the origin can be expanded uniquely as

$$F^{(\alpha,\beta)}(x, y) = \sum_{n=0}^{\infty} a_n R_n^{(\alpha,\beta)}(x, y), \qquad \alpha > \beta > -1/2 \tag{1}$$

in terms of the complete set

$$R_n^{(\alpha,\beta)}(x, y) = (x^2 + y^2)^n \, P_n^{(\alpha,\beta)}(x^2 - y^2/x^2 + y^2)/P_n^{(\alpha,\beta)}(1), \qquad n = 0, 1, 2,... \tag{2}$$

of biaxisymmetric harmonic polynomials. These even functions are classical solutions to the generalized biaxisymmetric potential equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{2\alpha + 1}{x}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + \frac{2\beta + 1}{y}\frac{\partial}{\partial y}\right) F^{(\alpha,\beta)} = 0 \tag{3}$$

subject to the Cauchy data

$$F_x^{(\alpha,\beta)}(0, y) = F_y^{(\alpha,\beta)}(x, 0) = 0$$

along the singular lines in $\Sigma^{(\alpha,\beta)}$.

Let us consider interpretations of this equation for some special values of the parameters α, β. If $2\alpha + 1$ and $2\beta + 1$ are non-negative integers, then the coordinates

$$x = (x_1{}^2 + \cdots + x_{2\alpha+2}^2)^{1/2}, \qquad y = (y_1{}^2 + \cdots + y_{2\beta+2}^2)^{1/2}$$

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interpret as a hypercircle on the intersection of hypercylinders through the point $(x_1, \ldots, x_{2\alpha+2}, y_1, \ldots, y_{2\beta+2}) \in E^{2(\alpha+\beta+2)}$, and Eq. (3) is equivalent to the biaxisymmetric Laplace equation. The limit $\alpha \downarrow \beta$ produces the generalized axisymmetric potential equation, and the zonal harmonics corresponding to (2),

$$R_n^{(\alpha,-1/2)}(x, y) = (x^2 + y^2)^n\, P_{2n}^{(\alpha,\alpha)}(x/(x^2 + y^2)^{1/2})/P_{2n}^{(\alpha,\alpha)}(1), \qquad n = 0, 1, \ldots,$$

after a quadratic transformation [1, p. 21], form a complete set for regular even generalized axisymmetric potentials (GASP). Reduction of the GASP equation to the harmonic equation E^2 follows from the lim $\alpha \downarrow -\frac{1}{2}$ that also reduces the zonal harmonics to the circular harmonics,

$$R_n^{(-1/2,-1/2)}(x, y) = (x^2 + y^2)^n \cos(2n \arccos(x/(x^2 + y^2)^{1/2})),$$

$n = 0, 1, 2, \ldots$ [1, *p*. 7] which analytically continue from the singular axis as the even polynomials

$$R_n^{(-1/2,-1/2)}(z, 0) = (-1)^n z^{2n}, \qquad n = 0, 1, 2, \ldots,$$

$z = x + iy \in \mathbf{C}$. These functions form complete sets for even harmonic, respectively analytic functions, regular at the origin. The GBASP, then, are natural extensions of harmonic or analytic functions. Hence, we anti cipate properties similar to those of the harmonic functions found from associated analytic f, by taking Ref, the real part of f. Concurrently, we seek operators analogous to Re.

The Bergman [2, 27] and Gilbert [6, 7] Integral Operator Method produced invertible operators generalizing Ref that successfully extend a variety of coefficient properties of analytic functions. These relate the growth of the Fourier coefficients of GBASP to the classification of its singularities, zeros, and extrema, creating analogies of the Theorems of Hadamard [4, 6], Caratheodory–Toeplitz [14, 25] and Caratheodory–Fejer [10, 15–17]. Essentially local information results. For the entire function GASP, the coefficients also provide global information. The growth characteristics of order and type of GASP, defined from the maximum modulus as in function theory [11], can be computed from the Fourier coefficients [5, 8]. As in the classical Markusevic–Gel'fond theorem [12], they are requisite in the construction of complete families of harmonic polynomials which approximate entire function GASP uniformly on simply connected axisymmetric compact sets [5]. For harmonic polynomial interpolation of GASP, the reader is referred to [13].

Analytic function theory also develops properties of f from the growth of various polynomial approximates measured in the Chebyshev sense. The theorem of S. N. Bernstein [3] identifies real entire analytic functions via such approximates and extends to determine the (positive finite) order and type by formulas due to R. S. Varga [26]. These techniques were incorporated with the Integral Operator Method in [18] to identity an entire function GASP from the convergence rate of the best axisymmetric harmonic polynomial approximates in the Chebyshev norm (c-norms). This characterizes a GASP, regular in an open hypersphere and continuous on its closure, that harmonically continues as an entire function GASP. Additionally, the order and type are defined explicitly as functions of the convergence rate of the c-norms. These appear analogous to Gilbert's coefficient formulas [8] for order and type. Also, they mirror the classifications of R. S. Varga [26] for entire analytic functions by relating the accuracy of local polynomial approximation of a GASP to its global existence and growth.

The preceding ideas extend in several directions. From the potential theoretic aspect, the order and type of an entire function GBASP can be defined to include those with zero or infinite order and transcendental GBASP with zero type. Order and type are then computed from the Fourier coefficients as in the formulas of A. R. Reddy [21–24] which generalize the growth-coefficient characterizations for analytic functions of finite order and type. Convergence properties of the c-norms, defined as in the axisymmetric case, measuring the error in the best harmonic polynomial approximates to real GBASP regular in an open hypersphere and continuous on its closure, identify those that exist globally and apply to the calculation of order and type as in analytic function theory [21–23]. These generalizations of the theory of analytic functions lead from local approximation properties to global characterization of solutions to the GASP equation.

The principal vehicle is an integral operator (and inverse) essentially developed in [16] which is an isometry between even analytic functions of one complex variable and GBASP on suitable domains of definition. We briefly list the properties of the operator and inverse in

2. Basic Formula

Let the operator mapping unique *associated* even analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^{2n}, \qquad z = x + iy \in \mathbf{C} \tag{4}$$

onto GBASP

$$F^{(\alpha,\beta)}(x, y) = \sum_{n=0}^{\infty} a_n R_n^{(\alpha,\beta)}(x, y), \qquad \alpha > \beta > -1/2 \tag{5}$$

be defined as in [15–16] from Koornwinder's integral for Jacobi polynomials [1, p. 32] as

$$
\begin{aligned}
F^{(\alpha,\beta)}(x, y) &= \mathscr{K}_{\alpha,\beta}(f) = \int_0^1 \int_0^\pi f(\zeta)\, d\mu_{\alpha,\beta}(t, s) \\
\zeta^2 &= x^2 - y^2t^2 + i\,2xyt \cos s \\
d\mu_{\alpha,\beta}(t, s) &= \gamma_{\alpha,\beta}(1 - t^2)^{\alpha-\beta-1}\, t^{2\beta+1}(\sin s)^{2\alpha}\, dt\, ds \\
\gamma_{\alpha,\beta} &= 2\Gamma(\alpha + 1)/\Gamma(1/2)\, \Gamma(\alpha - \beta)\, \Gamma(\beta + 1/2).
\end{aligned}
\tag{6}
$$

The inverse operator applies orthogonality of the Jacobi polynomials [1, p. 8] and the Poisson kernel [1, p. 11] to uniquely define the transform

$$
\begin{aligned}
f(z) &= \mathscr{K}^{-1}_{\alpha,\beta}(F^{(\alpha,\beta)}) = \int_{-1}^{+1} F^{(\alpha,\beta)}(r\xi, r(1 - \xi^2)^{1/2})\, d\nu_{\alpha,\beta}(z^2/r^2, \xi) \\
d\nu_{\alpha,\beta}(\tau, \xi) &= S_{\alpha,\beta}(\tau, \xi)(1 - \xi)^\alpha\, (1 + \xi)^\beta\, d\xi
\end{aligned}
\tag{7}
$$

whose kernel is written with the aid of (1, p. 12) in closed form as

$$
\begin{aligned}
S_{\alpha,\beta}(\tau, \xi) &= \eta_{\alpha,\beta} \frac{(1 - \tau)}{(1 + \tau)^{\alpha+\beta+2}} \\
&\quad \times {}_2F_1\left(\frac{\alpha + \beta + 2}{2}; \frac{\alpha + \beta + 3}{2}; \beta + 1; \frac{2\tau(1 + \xi)}{(1 + \tau)^2}\right) \\
\eta_{\alpha,\beta} &= \Gamma(\alpha + \beta + 2)/2^{\alpha+\beta+1}\, \Gamma(\alpha + 1)\, \Gamma(\beta + 1).
\end{aligned}
$$

The measures are normalized so that $\mathscr{K}^{-1}_{\alpha,\beta}(1) = \mathscr{K}_{\alpha,\beta}(1) = 1$. The *Envelope Method* [6, 7] easily establishes that the GBASP $F^{(\alpha,\beta)}$ is regular in the hypersphere $\Sigma_R^{(\alpha,\beta)}$: $x^2 + y^2 < R^2$ if, and only if its associate f is analytic in the disk $D_R : x^2 + y^2 < R^2$. On the singular axis $y = 0$, the identity

$$
f(x + i0) = F^{(\alpha,\beta)}(x, 0), \qquad |\, x \,| < R \tag{8}
$$

can be analytically continued as

$$
f(z) = F^{(\alpha,\beta)}(z, 0), \qquad |\, z \,| < R
$$

via the Law of Permanence of Functional Equations to recover the associate. These facts are summarized in

THEOREM 1. *For each GBASP $F^{(\alpha,\beta)}$ regular in the hypersphere $\Sigma_R^{(\alpha,\beta)}$ there is a unique $\mathscr{K}_{\alpha,\beta}$ associated even function f analytic in the disk D_R and conversely.*

Having established these basic formulas, we consider our first objective which is

3. Growth of Entire Function GBASP

The maximum moduli of GBASP and associate are defined as in complex function theory [21, p. 129, 132],

$$m(r, f) = \max_{|z|=r} |f(z)|$$

$$M(r, F^{(\alpha,\beta)}) = \max_{x^2+y^2=r^2} |F^{(\alpha,\beta)}(x, y)|$$

as are the *upper* and *lower orders*

$$\begin{matrix} P(k,j) \\ \Lambda(k,j) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{l_{k+j} M(r, F^{(\alpha,\beta)})}{l_{j+1} r}$$

$$\begin{matrix} \rho(k,j) \\ \lambda(k,j) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{l_{k+j} m(r, f)}{l_{j+1} r}$$

and *upper* and *lower types*

$$\begin{matrix} T(k) \\ \Omega(k) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{l_k M(r, F^{(\alpha,\beta)})}{r^{P(k)}},$$

$$\begin{matrix} \tau(k) \\ \omega(k) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{l_k m(r, f)}{r^{\rho(k)}}$$

[21, p. 129] with $l_k x = \log\log \cdots$ (k times) x.
The *classification of index* is k if $\rho(k-1) = \infty$, $\rho(k) < \infty$ where

$$P(k, 0) = P(k) \qquad P(2, 0) = P$$

$$\rho(k, 0) = \rho(k)\ \rho(2, 0) = \rho$$

with logarithmic order ($\rho = 0$)

$$\rho(1, 1) = \rho_l\,, \qquad \rho(1, 1) = \rho_l$$

and logarithmic type ($1 < \rho_1 < \infty$)

$$\begin{matrix} T_l \\ \Omega_l \end{matrix} = \lim_{r\to\infty} \inf \frac{l_2 M(r, F^{(\alpha,\beta)})}{l_2 r}$$

$$\begin{matrix} \tau_l \\ \omega_l \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{l_2 m(r, f)}{l_2 r}.$$

The essential equivalence of corresponding quantities for GBASP and associate is considered in

THEOREM 2. *Let $F^{(\alpha,\beta)}$ be a real valued entire function GBASP with $\mathscr{K}_{\alpha,\beta}$ associate f. Then the upper and lower orders and upper and lower types of $F^{(\alpha,\beta)}$ and f respectively are identical. This is true of the logarithmic orders and types as well.*

Proof. Let us consider the relation for $F^{(\alpha,\beta)}$,

$$F^{(\alpha,\beta)}(x, y) = \mathscr{K}_{\alpha,\beta}(f)$$

defined globally by Theorem 1. The non-negativity and the normalization of the measure lead directly to the bound

$$M(r, F^{(\alpha,\beta)}) \leqslant m(r, f)$$

and the consequent inequality

$$\frac{l_{k+j}M(r, F^{(\alpha,\beta)})}{l_{j+1}r} \leqslant \frac{l_{k+j}m(r, f)}{l_{j+1}r}. \tag{9}$$

The inverse relation,

$$f(z) = \mathscr{K}^{-1}_{\alpha,\beta}(F^{(\alpha,\beta)}).$$

also valid globally by Theorem 1, leads to the estimates

$$|f(z)| \leqslant M(r, F^{(\alpha,\beta)})\, N_{\alpha,\beta}(\tau), \qquad \tau = (zr^{-1})^2$$

and

$$N_{\alpha,\beta}(\tau) = \max\{\eta^{-1}_{\alpha,\beta} \mid S_{\alpha,\beta}(\tau, \xi)|; -1 \leqslant \xi \leqslant +1\}.$$

However, for $z = \epsilon re^{i\theta}$ (ϵ real)

$$m(\epsilon r, f) \leqslant M(r, F^{(\alpha,\beta)})\, N_{\alpha,\beta}(\tau)$$

gives

$$m(r, f) \leqslant M(\epsilon^{-1}r, F^{(\alpha,\beta)})\, N_{\alpha,\beta}(\epsilon^2)$$

and

$$\frac{l_{k+j}m(r, f)}{l_{j+1}r} \leqslant \frac{l_{k+j-1}(l_1 M(\epsilon^{-1}r, F^{(\alpha,\beta)}) + l_1 N_{\alpha,\beta}(\tau))}{l_{j-1}(l(\epsilon^{-1}r) - l(\epsilon^{-1}))}. \tag{10}$$

From the inequalities (9) and (10), the requisite conclusions concerning upper and lower orders follow directly. Using equality of the orders, it is apparent that similar reasoning establishes the asserted equality of the upper and lower types.

Having established equality of the respective *upper* and *lower orders* and *types* of GBASP and associate, we shall henceforth indicate them in the lower case symbols. We remark that the classifications are necessarily independent of the dimension of the space (parameters (α, β)) containing the domain of the GBASP because of the normalization. For the same reason, this independence of (α, β) is effectively the case for the developments concerning approximation. As typical examples of Theorem 2 we consider the following

COROLLARY 2.1. *Necessary and Sufficient conditions that the GBASP*

$$F^{(\alpha,\beta)}(x, y) = \sum_{n=0}^{\infty} a_n R_n^{(\alpha,\beta)}(x, y)$$

be an entire function of

(i) *index k and order $\rho(k)$ is*

$$\lim \sup(nl_k n/\log(1/| a_n |)) = \rho(k)$$

or

(ii) *index k with order $\rho = \rho(k) > 0$ and type $\tau(k)$ are*

$$\lim \sup \frac{n}{\rho e} | a_n |^{\rho/n} = \tau(k)$$

and

$$\lim \sup(l_{k-1}n) | a_n |^{\rho/n} = \tau(k)$$

or

(iii) *logarithmic order ρ_l and type τ_l are*

$$\lim \sup(l_1 n/l(-1/nl_1(| a_n |))) = \rho_l - 1$$

and

$$\lim \sup \frac{(n/\rho_l)^{\rho_l}}{\left(-\dfrac{l_1 | a_n |}{\rho_l - 1}\right)^{\rho_l - 1}} = \tau_l .$$

Proof. By Theorem 1, $F^{(\alpha,\beta)}$ is entire if, and only if, the associate is entire. Moreover, the order and type of the associate agree. Consequently, to prove; (i) we cite [21, p. 130] Lemma 1, (ii) [21, p. 130] Lemma 3 and (iii) [21, p. 131] Lemma 5 and 7. Note that (ii) is the biaxisymmetric version of Gilbert's transplant theorem [6, p. 188] of [11; 4, p. 293] when index $k = 1$.

From the previous reasoning, it appears that the bulk of references [21–23] concerning the relations among orders and types of analytic functions and their Taylor's coefficients extends to GBASP quid pro quo. We shall not do this verbatim, but cite the above result as representative. This same philosophy applies to the interrelations of order and type that are next considered in

4. Polynomial Approximation

Let the Chebyshev norms [21, p. 128] (c-norms) be defined for $f \in C([-1, +1])$ and $F^{(\alpha,\beta)} \in C(\partial\Sigma^{(\alpha,\beta)})$ as

$$\begin{aligned} e_n(f) &= \inf\{\| f - p \|_* \,;\, p \in \hbar_n\} \qquad n = 0, 1, 2,\ldots \\ \| f - p \|_* &= \sup_{x\in[-1,1]} | f(x) - p(x)| \end{aligned} \tag{11}$$

and

$$\begin{aligned} E_n(F^{(\alpha,\beta)}) &= \inf\{\| F^{(\alpha,\beta)} - P^{(\alpha,\beta)} \|;\, P^{(\alpha,\beta)} \in \mathscr{H}_n^{(\alpha,\beta)}\}, \qquad n = 0, 1, 2,\ldots \\ \| F^{(\alpha,\beta)} - P^{(\alpha,\beta)} \| &= \sup_{x^2+y^2=1} | F^{(\alpha,\beta)}(x, y) - P^{(\alpha,\beta)}(x, y)|. \end{aligned} \tag{12}$$

The set $\hbar_n$ contains all real polynomials of degree atmost $2n$, and the set $\mathscr{H}_n^{(\alpha,\beta)}$ contains all real biaxisymmetric harmonic polynomials of degree atmost $2n$. The operators $\mathscr{K}_{\alpha,\beta}$ and $\mathscr{K}_{\alpha,\beta}^{-1}$ establish one-one equivalence of the sets $\hbar_n$ and $\mathscr{H}_n^{(\alpha,\beta)}$. The existence of GBASP globally and growth of the c-norms E_n is taken up in

Theorem 3. *Let $F^{(\alpha,\beta)}$ be a real valued GBASP regular in $\Sigma^{(\alpha,\beta)}$ and continuous on $\bar{\Sigma}^{(\alpha,\beta)}$ and k be a positive integer. Then both*

$$\limsup nE_n^{1/n}(F^{(\alpha,\beta)}) = (\rho e\tau)\, 2^{-\rho}, \qquad k > 1$$

and

$$\limsup (l_{k-1}n)\, E_n^{\rho/n}(F^{(\alpha,\beta)}) = \tau(k)\, 2^{-\rho}$$

are finite if, and only if, $F^{(\alpha,\beta)}$ has an analytic continuation as an entire function GBASP of index k, with order $\rho(k) > 0$ and type $\tau(k)$ finite.

Proof. Let the real GBASP $F^{(\alpha,\beta)}$ be regular in $\Sigma^{(\alpha,\beta)}$ and continuous on $\bar{\Sigma}^{(\alpha,\beta)}$. Let the above listed limits be satisfied by the polynomial approximations of $F^{(\alpha,\beta)}$ on $\partial\Sigma^{(\alpha,\beta)}$ relative to the c-norm. The associate f is analytic in D and continuous at $x = \pm 1$ where it coincides with $F^{(\alpha,\beta)}$ so that application of the maximum principle to Eq. (8) gives

$$| f(x) - p(x)| = | F^{(\alpha,\beta)}(x, 0) - P^{(\alpha,\beta)}(x, 0)| \leqslant \| F^{(\alpha,\beta)} - P^{(\alpha,\beta)} \|,$$
$$| x | \leqslant 1 \quad \text{for} \quad p \in \hbar_n \quad \text{and} \quad P^{(\alpha,\beta)} = \mathscr{K}_{\alpha,\beta}(p).$$

Then

$$e_n(f) \leqslant \| F^{(\alpha,\beta)} - P^{(\alpha,\beta)} \|, \qquad P^{(\alpha,\beta)} \in \mathscr{H}_n^{(\alpha,\beta)}. \tag{13}$$

For each given $\epsilon > 0$, there correspond $m_1 = m_1(\epsilon)$, subsequence $n_j = n_j(\epsilon)$ and polynomials $P^{(\alpha,\beta)} \in \mathscr{H}_{n_j}^{(\alpha,\beta)}$ such that

$$E_n(F^{(\alpha,\beta)}) - \epsilon \leqslant \| F^{(\alpha,\beta)} - P^{(\alpha,\beta)} \| \leqslant E_{n_j}(F^{(\alpha,\beta)}) + \epsilon \tag{14}$$

for index $n_j > m_1(\epsilon)$. Combining these gives

$$e_n(f) \leqslant E_{n_j}(F^{(\alpha,\beta)}) + \epsilon. \tag{15}$$

On the assumption concerning the value of first limit there is an $m_2 = m_2(\epsilon)$ such that for $n \geqslant m_2$,

$$E_n(F^{(\alpha,\beta)}) \leqslant \left(\frac{(\rho e \tau)\, 2^{-\rho} + \epsilon}{n}\right)^n, \tag{16}$$

a quantity ultimately less than 1 for some integer $m_3 = m_3(\epsilon)$. Then for $n \geqslant \max\{m_1, m_2, m_3\}$, we have

$$n e_n^{\rho/n}(f) \leqslant n_j E_{n_j}^{\rho/n_j}(F^{(\alpha,\beta)}) + \epsilon$$

and

$$(l_{k-1} n)\, e_n^{\rho/n}(f) \leqslant (l_{k-1} n_j)\, E_{n_j}^{\rho/n_j}(F^{(\alpha,\beta)}) + \epsilon.$$

Also the inequalities

$$\limsup n e_n^{\rho/n}(f) \leqslant \limsup n E_n^{\rho/n}(F^{(\alpha,\beta)})$$

and

$$\limsup (l_{k-1} n)\, e_n^{\rho/n}(f) \leqslant \limsup (l_{k-1} n)\, E_n^{\rho/n}(F^{(\alpha,\beta)})$$

hold, identifying the associate f (see [21, p. 134]) as an entire function of order at most $\rho(k)$. From this fact and the normalization of the $\mathscr{K}_{\alpha,\beta}$ transform, it follows that

$$\| F^{(\alpha,\beta)} - P^{(\alpha,\beta)} \| \leqslant \sup_{|z| \leqslant 1} |f(z) - p(z)| \equiv \| f - p \|,$$

and from inequality (14) it follows that

$$E_n(F^{(\alpha,\beta)}) - \epsilon \leqslant e_n^*(f) \tag{17}$$

for $n \geqslant m_1(\epsilon)$ and

$$e_n^*(f) \equiv \inf\{\| f - p \|;\, p \in \mathscr{h}_n\}, \qquad n = 0, 1, 2, \ldots .$$

To compare the norms e^* and e_n, expand the even entire function f in a Chebyshev series [20, p. 91] as

$$f(z) = \alpha_0 + 2 \sum_{n=1}^{\infty} \alpha_n T_n(z^2).$$

The extremal polynomial $p \in \mathscr{h}_N$ has the expansion

$$p(z) = \alpha_0 + 2 \sum_{n=1}^{N} \alpha_n T_n(z^2).$$

In the ellipse $\mathscr{E}_\delta : | z - 1 | + | z + 1 | \leqslant 2\delta$ $(\delta > 1)$, the inequalities

$$\begin{aligned} e_N(f) \leqslant e_N^*(f) \leqslant \| f - p \| &\leqslant 2 \sum_{n=N+1}^{\infty} (| \alpha_n | \sup_{\delta} | T_n(z^2)|) \\ &\leqslant 2M(\delta^2 + 1/2\delta, f)\, e^{\epsilon\delta}/\delta^{2N}(\delta - 1) \end{aligned}$$

hold since the Chebyshev polynomials T_n are entire functions of polynomial growth. Thus

$$0 \leqslant e_N(f) \leqslant e_N^*(f) \leqslant 2M(\delta, f)\, e^{\epsilon\delta}/\delta^{2N}(\delta - 1)$$

and

$$\lim(e_n^*(f) - e_n(f)) = 0. \tag{18}$$

In view of Eq. (18), there is an $m_4 = m_4(\epsilon)$ such that

$$e_n^*(f) - \epsilon \leqslant e_n(f), \qquad n \geqslant m_4 . \tag{19}$$

Combining (17) and (19) gives

$$E_n(F^{(\alpha,\beta)}) \leqslant e_n(f) + 2\epsilon, \qquad n \geqslant \max\{m_1 , m_4\}. \tag{20}$$

Then

$$\limsup nE_n^{\rho/n}(F^{(\alpha,\beta)}) \leqslant n \limsup ne_n^{\rho/n}(f)$$

and

$$\limsup(l_{k-1}n)\, E_n^{\rho/n}(F^{(\alpha,\beta)}) \leqslant \limsup(l_{k-1}n)\, e_n^{\rho/n}(f).$$

Thus the associate meets the same limiting requirements as the GBASP. Consequently, (see Reddy [21, p. 134]) the associate is an entire function of index k, order $\rho(k)$ and type $\tau(k)$. The same is true of the GBASP by Theorem 2.

Conversely, let the real GBASP $F^{(\alpha,\beta)}$ regular in $\Sigma^{(\alpha,\beta)}$ and continuous on $\bar{\Sigma}^{(\alpha,\beta)}$ have extension as an entire GBASP of index k, order $\rho(k)$ and type

$\tau(k)$. This is also true of the associate, which then meets the requisite limits [21, p. 134]

$$\limsup n e^{\rho/n}(f) = (\rho e \tau)\, 2^{-\rho}, \qquad k > 1$$

and

$$\limsup (l_{k-1} n)\, e_n^{\rho/n}(f) = \tau(k)\, 2^{-\rho}.$$

Since the associate clearly meets the analyticity and regularity requirements of the first part of the proof, the previous reasoning and estimates apply and establish those limits listed in the theorem.

As an application, the following analogy of the S. N. Bernstein theorem is considered.

COROLLARY 3.1. *Let the real valued GBASP $F^{(\alpha,\beta)}$ be regular in $\Sigma^{(\alpha,\beta)}$ and continuous on $\bar{\Sigma}^{(\alpha,\beta)}$. Then a necessary and sufficient condition that $F^{(\alpha,\beta)}$ be the restriction of an entire function GBASP is that*

$$\limsup E_n^{1/n}(F^{(\alpha,\beta)}) = 0.$$

Proof. If the above limit condition is satisfied by $F^{(\alpha,\beta)}$ in the hypothesis, inequality (15) gives

$$\lim e_n^{1/n}(f) \leqslant \limsup E_n^{1/n}(F^{(\alpha,\beta)}) = 0.$$

By the classical Bernstein theorem [3; 23, p. 176] f is entire and hence so is $F^{(\alpha,\beta)}$. Conversely, if $F^{(\alpha,\beta)}$ is entire, so is f and (20), the reverse to (15), holds. Then application of (20) gives the inequality

$$\limsup E_n^{1/n}(F^{(\alpha,\beta)}) \leqslant \lim e_n^{1/n}(f)$$

so that [23, p. 176] completes the reasoning. As a second application we consider

COROLLARY 3.2. *Let the real valued GBASP $F^{(\alpha,\beta)}$ be regular in $\Sigma^{(\alpha,\beta)}$ and continuous on $\bar{\Sigma}^{(\alpha,\beta)}$. Then a necessary and sufficient condition that $F^{(\alpha,\beta)}$ be the restriction of an entire function GBASP of order <1 or of order 1 and type 0 is*

$$\lim n E_n^{1/n}(F^{(\alpha,\beta)}) = 0.$$

Proof. Let $F^{(\alpha,\beta)}$, as in the hypothesis, be the restriction of an entire *GBASP* of order ρ and type τ; with Theorem 3 we have shown that

$$\limsup \frac{n}{\rho e} E_n^{\rho/n}(F^{(\alpha,\beta)}) = \tau\, 2^{-\rho}. \tag{21}$$

Reasoning from Theorem 4 produces the analogy of the classical Varga [26] result in

$$\limsup (n l_1 n / l_1(1/E_n(F^{(\alpha,\beta)}))) = \rho$$

as the order of $F^{(\alpha,\beta)}$ and hence

$$\limsup (n l_1 n / l_1(1/e_n(f))) = \rho$$

as the order of the associate. Thus given $\epsilon > 0$ and $n(\epsilon) > 0$ we find as in Theorem 1 [22, p. 101] that

$$nE_n^{1/n}(F^{(\alpha,\beta)}) \leqslant n^{1-1/\rho+\epsilon} \tag{22}$$

for $n \geqslant \max\{m_1, m_2, m_4\}$. Thus, if $\rho < 1$,

$$\lim nE_n^{1/n}(F^{(\alpha,\beta)}) = 0.$$

If $\rho = 1$ and $\tau = 0$, we reason by the first equality in the statement of Theorem 3 that

$$\lim nE_n^{1/n}(F^{(\alpha,\beta)}) = 0$$

and then

$$\limsup E_n^{1/n}(F^{(\alpha,\beta)}) = 0$$

which completes the proof by the previous corollary.

The next result is the potential-theoretic transplant of Varga's generalized theorem [26, p. 132]

THEOREM 4. *Let the real valued GBASP $F^{(\alpha,\beta)}$ be regular in $\Sigma^{(\alpha,\beta)}$ and continuous on $\bar{\Sigma}^{(\alpha,\beta)}$. Then $F^{(\alpha,\beta)}$ is the restriction to $\bar{\Sigma}^{(\alpha,\beta)}$ of an entire function GBASP of order $\rho(k) = \sigma$ if, and only if,*

$$\limsup \frac{n l_k n}{l_1(1/E_n(F^{(\alpha,\beta)}))} = \sigma.$$

Proof. Let $F^{(\alpha,\beta)}$ satisfy the hypothesis and the stated limit condition. Let $\epsilon > 0$ be given, use (15) and monotonicity of l_1 to give

$$l_1(1/E_{n_j}(F^{(\alpha,\beta)}) + \epsilon) \leqslant l_1(1/e_n(f)) \tag{23}$$

as in Theorem 3. From (16), $\lim E_n < 1$ so that the smaller term in (23) is positive for all n_j sufficiently large. Then for large $n_j > n$,

$$\frac{n l_1 n}{l_1(1/\epsilon_n(f))} \leqslant \frac{n_j l_k n_j}{l_1(1/E_{n_j}(F^{(\alpha,\beta)}) + \epsilon)} \tag{24}$$

However,

$$\sigma = \limsup \frac{n_j l_k n_j}{l_1(1/E_{n_j}(F^{(\alpha,\beta)}) + \epsilon)}$$

so that from (24)

$$\frac{n l_k n}{l_1(1/e_n(f))} \leqslant \sigma + \epsilon$$

for *all* indices n sufficiently large. For the reverse, use (20),

$$l_1(1/e_n(f) + 2\epsilon) \leqslant l_1(1/E_n(F^{(\alpha,\beta)}))$$

for all n sufficiently large. Then

$$\sigma - \epsilon \leqslant \frac{n l_k n}{l_1(1/E_n(F^{(\alpha,\beta)}))} \leqslant \frac{n l_k n}{l_1(1/e_n(f) + 2\epsilon)}$$

with the smaller inequality holding for infinitely many indices. As above

$$\limsup \frac{n l_k n}{l_1(1/e_n(f) + 2\epsilon)} = \limsup \frac{n l_k n}{l_1(1/e_n(f))}$$

so that

$$\sigma - \epsilon \leqslant \frac{n l_k n}{l_1(1/e_n(f))}$$

for infinitely many indices. Thus,

$$\sigma = \limsup \frac{n l_k n}{l_1(1/e_n(f))},$$

the order of entire function f. *GBASP* $F^{(\alpha,\beta)}$ is an entire function of the same order. For brevity, the proof of the converse is omitted. We observe that for $k = 1$, this formula is the biaxisymmetric version of the result in [18].

Our final application is with respect to *GBASP* with logarithmic order.

THEOREM 5. *Let real valued GBASP $F^{(\alpha,\beta)}$ be regular in $\Sigma^{(\alpha,\beta)}$ and continuous on $\bar{\Sigma}^{(\alpha,\beta)}$. Then $F^{(\alpha,\beta)}$ is the restriction to $\bar{\Sigma}^{(\alpha,\beta)}$ of an entire function GBASP of logarithmic order $\rho_l = 1 + \alpha$ if, and only if,*

$$\limsup \frac{l_1 n}{l_1(1/n l_1(1/E_n(F^{(\alpha,\beta)})))} = \alpha.$$

Proof. Let real value GBASP $F^{(\alpha,\beta)}$ described in the hypothesis satisfy the asserted limit condition. From inequalities (15) and (16) we find

$$l_1(1/n_j l_1(1/E_{n_j}(F^{(\alpha,\beta)}) + \epsilon)) \leqslant l_1(1/n l_1(1/e_n(f)))$$

for $n_j > n$ sufficiently large. Then

$$\frac{l_1 n}{l_1(1/n l_1(1/e_n(f)))} \leqslant \frac{l_1 n_j}{l_1(1/n_j l_1(1/E_{n_j}(F^{(\alpha,\beta)}) + \epsilon))}$$

as in the previous theorem, we deduce

$$\frac{l_1 n}{l_1(1/n l_1(1/e_n(f))} \leqslant \alpha + \epsilon$$

for all but finitely many indices. From (20) and the specified limit, one easily finds

$$\alpha - \epsilon \leqslant \frac{l_1 n}{l_1(1/n l_1/E_n(F^{(\alpha,\beta)}))} \leqslant \frac{l_1 n}{l_1(1/n l_1(1/e_n(f) + 2\epsilon))}$$

for infinitely many indices. Thus

$$\alpha - \epsilon \leqslant \frac{l_1 n}{l_1(1/n l_1(1/e_n(f)))} \leqslant \alpha + \epsilon$$

with the smaller inequality valid for infinitely many indices and the larger for all but finitely many. Consequently, the associate is an entire function of logarithmic order $\rho_l = 1 + \alpha$ (see [21, p. 135]). The same holds of $F^{(\alpha,\beta)}$. The converse proof is similar.

4. Remarks Suggesting Further Applications

To consider GBASP $R_n^{(\alpha,\beta)}$ with parameters $\beta > \alpha$, refer to the symmetry relation [1, p. 8] for Jacobi polynomials. The limit of $R_n^{(\alpha,\beta)}$ as $\alpha \downarrow \beta$ indicates the GBASP form of the results in sections 1–3.

The notions of *regular growth* and *perfectly regular growth* [22, p. 99–100] are utilized by Reddy [22, p. 104ff.] to study the relative growth of c-norms $e_n(f)$ and Taylor coefficients a_n. Adopting the function theory definitions for GBASP, we can use the operators (6–7) to show that a GBASP is of regular/perfectly regular growth if, and only if, the associate is of regular/perfectly regular growth. This suggests comparisons of the relative growths of c-norms $E_n(F^{(\alpha,\beta)})$ and Fourier coefficient a_n. A similar study can be made of *proximate order*, a refinement of type introduced in classical function theory.

The Method of Ascent and the inverse Method of Descent (see [7]) construct maps between the linear space of harmonic functions and the linear space of solutions of more general elliptic partial differential equations. Direct generalization of the preceding is possible by composition of operators for those biaxisymmetric equations whose lower order derivatives have coefficients that permit construction of operators that are norm preserving maps between the linear space of entire function GBSAP and entire function solutions. Equations whose coefficients do not have this property suggest upper and lower estimates on the order and type and necessary or sufficient conditions for the existence of solutions as entire functions.

Concerning those potentials that are not entire functions, the question arises as to what information can be found from local approximates about properties of the singularities. Chebyshev approximation of GASP that are regular in a closed hypersphere by linear combinations of axisymmetric harmonic polynomials and Newtonian potentials leads to characterization of the singularities of the principal branch of the harmonic continuation of the GASP [19].

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